Advanced Calculus

Theorem Sheet

Axioms A1 (Basic Properties of $\mathbb{R}$):

1. Closure of addition and multiplication
2. Commutativity of addition
3. Associativity of addition
4. Existence of an additive identity
5. Existence of an additive inverse
6. Commutativity of multiplication
7. Associativity of multiplication
8. Existence of a multiplicative identity
9. Existence of multiplicative inverses
10. The Distributive Property
11. The Nontriviality Assumption

Theorems T2 (Basic Properties of $\mathbb{R}$):

1. The additive identity, 0 is unique.
2. $a0 = 0a = 0$
3. $ab = 0 \Rightarrow a = 0$ or $b = 0$
4. The equation $a + x = 0$ has a solution.
5. The solution to the above equation is unique.
6. The multiplicative identity is unique.
7. $a \neq 0 \Rightarrow ax = 1$ has a solution.
8. The solution to the above equation is unique.
9. $-(-a) = a$
10. $a \neq 0 \Rightarrow (a^{-1})^{-1} = a$
11. $a \neq 0 \Rightarrow (-a^{-1}) = -a^{-1}$

Axioms A3 (Positivity Axioms):

1. a, b are positive $\Rightarrow ab$ and $a + b$ are positive.
2. Exactly one of the following is true
   - a is positive
   - $-a$ is positive
   - $a = 0$
3. $a > b$ means $a - b$ is positive.
4. $a > 0$ means a is positive
5. $a \geq b$ means $a - b$ is positive or zero.

Theorems T4 (Positivity Properties):

1. $a \neq 0 \Rightarrow a^2 > 0$
2. $1 > 0$
3. $a > 0 \Rightarrow a^{-1} > 0$
4. $c > 0$ and $a > b \Rightarrow ac > bc$
5. $c < 0$ and $a > b \Rightarrow ac < bc$

Theorems T5 (Induction Theorems):

1. Theorem: $\mathbb{N}$ is inductive
2. If $A \subseteq \mathbb{N}$ is inductive, then $A = \mathbb{N}$.
3. Let $S(n)$ be a statement (claim) based on the natural number n. Assume the following are true:
   - $S(1)$
   - $S(k) \Rightarrow S(k+1)$

   Then $S(n)$ is true for every natural number n.

Theorems T6 (Theorems on numbers):

1. $n, m \in \mathbb{N} \Rightarrow n + m \in \mathbb{N}$
2. $n, m \in \mathbb{N} \Rightarrow nm \in \mathbb{N}$
3. If $x \in \mathbb{Q}$, then there are some $m, n \in \mathbb{Z}$ with at least one of them odd such that $x = \frac{m}{n}$
4. If $n \in \mathbb{Z}$ is even, then n^2 is as well.

Axioms A7 (Sup exists): Every set of real numbers that has an upper bound, has a single smallest upper bound.

Theorem T8 ($\sqrt{x}$ exists): Let c be a positive number. There is a unique solution to the system below.

$$x > 0$$
$$x^2 = c$$

Theorems T9 (Archimedean Property):

1. $\forall_{c>0} \exists_{n \in \mathbb{N}} (n > c)$
2. $\forall_{\varepsilon>0} \exists_{n \in \mathbb{N}} \left(\frac{1}{n} < \varepsilon\right)$

Theorem T10: Let $n \in \mathbb{Z}$. There is no integer in the interval $(n, n+1)$

Theorem T11: Assume $\emptyset \neq S \subseteq \mathbb{Z}$, and that S is bounded above. Then S has a maximum element.

Theorem T12: $\forall_{c \in \mathbb{R}} \exists!_{k \in \mathbb{Z}} \left(k \in [c, c+1)\right)$

Theorem T13: $\mathbb{Q}$ is dense in $\mathbb{R}$.

Theorems T14: For $x \in \mathbb{R}$, $d > 0$:

1. $|x| \leq d$ iff $-d \leq x \leq d$
2. $-|x| \leq x \leq |x|$

Theorem T15 (The Triangle Inequality): For all real a, b: $|a + b| \leq |a| + |b|$

Theorem T16 (The Reverse Triangle Inequality): For all real a, b: $\big||a| - |b|\big| < |a - b|$

Theorem T17: Fix $a \in \mathbb{R}$ and $r > 0$. TFAE:

- $|x - a| < r$
- $a - r < x < a + r$
- $x \in (a - r, a + r)$

Theorem T18: Let $a, b \in \mathbb{R}$, $n \in \mathbb{N}$. Then:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + \cdots + b^{n-1})$$

$$a^n - b^n = (a - b)\sum_{k=0}^{n-1} a^{(n-1)-k} b^k$$

Theorem T19 (Finite geometric series): Let $m \in \mathbb{N}; r \neq 1$. Then:

$$1 + r + r^2 + \cdots + r^m = \frac{1 - r^{m+1}}{1 - r}$$

$$\sum_{k=0}^{m} r^k = \frac{1 - r^{m+1}}{1 - r}$$

Theorem T20 (Binomial Theorem): $a, b \in \mathbb{R},\ n \in \mathbb{N}$. Then:

$$(a + b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \cdots + \binom{n}{n} b^n$$

$$(a + b)^n = \sum_{k=0}^{n} \binom{n}{k} a^{n-k} b^k$$

Lemma L21: If $\{a_n\} \to 0$ and $\exists_{N\in\mathbb{N}} \forall_{n \geq N} (|b_n| \leq C|a_n|)$ then also $\{b_n\} \to 0$.

Lemma L22: If $\{a_n\} \to a$ and $\exists_{N\in\mathbb{N}} \forall_{n \geq N} (|b_n - b| \leq C|a_n - a|)$ then also $\{b_n\} \to b$.

Theorem T23 (Sum property for convergence): Assume $\{a_n\} \to a$ and $\{b_n\} \to b$. Then $\{a_n + b_n\} \to a + b$.

Lemma L24: Assume $\{a_n\} \to a$, then $\{ca_n\} \to ca$.

Lemma L25: Assume $\{a_n\} \to 0$ and $\{b_n\} \to 0$, then also $\{a_n b_n\} \to 0$.

Theorem T26 (product property for convergence): Assume $\{a_n\} \to a$ and $\{b_n\} \to b$. Then $\{a_n b_n\} \to ab$.

Theorem T27: Assume $b_n \neq 0,\ b \neq 0$, and $\{b_n\} \to b$. Then $\left\{\frac{1}{b_n}\right\} \to \frac{1}{b}$.

Theorem T28 (Quotient property for convergence): Assume $b_n \neq 0$, $b \neq 0$, $\{a_n\} \to a$, and $\{b_n\} \to b$. Then $\left\{\frac{a_n}{b_n}\right\} \to \frac{a}{b}$

Theorem T29 (Linearity property of convergence): Assume $\{a_n\} \to a$, and $\{b_n\} \to b$. Then $\{ca_n + db_n\} \to ca + db$

Theorem T30 (Polynomial property for convergence): Assume $\{a_n\} \to a$, and $f(x)$ is a polynomial. Then the polynomial of the sequence also converges: $\{f(a_n)\} \to f(a)$

Theorem T31: Every convergent sequence is bounded.

Theorem T32: A set S is dense in $\mathbb{R}$ if and only if every real number is the limit of a sequence in S:

Lemma L33: Assume $\{d_n\} \to d$ and $d_n \geq 0$ for each n, then $d \geq 0$.

Theorem T34: Assume $\{c_n\} \to c$ and $c_n \in [a, b]$ for all $n \in \mathbb{N}$. Then $c \in [a, b]$

Theorem T35 (Monotone Convergence Theorem): Let $\{a_n\}$ be a monotone sequence. Then $\{a_n\}$ converges if and only if it is bounded. Furthermore, if it does converge, it converges to either its sup or inf.

Theorem T36 (Nested Interval Theorem): Construct a sequence of intervals $I_n \coloneqq [a_n, b_n]$ that are nested, by which we mean $\forall_{n\in\mathbb{N}}(I_{n+1} \subseteq I_n)$.
If $\{b_n - a_n\} \to 0$, then for some $c \in \mathbb{R}$:

$$\{a_n\} \to c$$
$$\{b_n\} \to c$$
$$\bigcap_{n=1}^{\infty} I_n = \{c\}$$

Theorem T37: Let $\{a_n\}$ be a sequence and assume $\{a_n\} \to a$. Then every subsequence also converges to a. That is, $\{a_{n_k}\} \to a$

Theorem T38: Every sequence has a monotone subsequence.

Theorem T39: Every bounded sequence has a convergent subsequence.

Theorem T40: **(Sequential Compactness of closed intervals):** $[a, b]$ is sequentially compact for all $a < b$.

Theorem T41: Let $S \subseteq \mathbb{R}$. The following are equivalent:
1. S is closed and bounded.
2. S is sequentially compact
3. S is compact

Theorem T42: Let $f, g: D \to \mathbb{R}$ both be continuous functions. Then $f + g, f - g$, and $f \cdot g$ are also continuous.

Theorem T43: Let $f, g: D \to \mathbb{R}$ both be continuous functions. Assume $g(x) \neq 0$ on D. Then $\frac{f}{g}$ is continuous.

Corollary C44: Let $p, q: \mathbb{R} \to \mathbb{R}$ be polynomials. Then p and q are continuous, as well as the rational function $\frac{p}{q}: D \to \mathbb{R}$ where $D = \{x \in \mathbb{R} | q(x) \neq 0\}$.

Theorem T45: Let $f: D \to \mathbb{R}$ and $g: U \to \mathbb{R}$. Assume the following.

- $f(D) \subseteq U$
- f is continuous at $x_0 \in D$
- g is continuous at $f(x_0) \in U$.

Then $g \circ f$ is continuous at x_0.

Lemma L46: Let $f: [a, b] \to \mathbb{R}$ be a continuous function. The image of f is bounded.

Theorem T47 (Extreme Value Theorem): Let $f: [a, b] \to \mathbb{R}$ be continuous. Then f attains both a maximum and minimum value.

Theorem T48 (Intermediate Value Theorem): Let $f: [a, b] \to \mathbb{R}$ be continuous. Let $c \in \mathbb{R}$ such that $f(a) < c < f(b)$. Then there is some $x_0 \in (a, b)$ such that $f(x_0) = c$. The same is true if we replace each "<" with ">".

Theorem T49: Let I be an interval and $f: I \to \mathbb{R}$ be continuous. Then $f(I)$ is also an interval.

Theorem T50: Let $f: D \to \mathbb{R}$ be a uniformly continuous function. Then f is also continuous.

Theorem T51: Let $f: [a, b] \to \mathbb{R}$ be continuous. Then f is also uniformly continuous.

Theorem T52: Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$. The sequential definition of continuity at $x_0 \in D$ is equivalent to the $\varepsilon - \delta$ criterion of continuity at x_0. Also, the uniformly continuous definition is equivalent to the $\varepsilon - \delta$ criterion of uniform continuity.

Theorem T53: Suppose $f: D \to \mathbb{R}$ is monotone. If $f(D)$ is an interval, then f is continuous.

Theorem T54: Let I be an interval and $f: I \to \mathbb{R}$ a monotone function. Then f is continuous iff its image $f(I)$ is an interval.

Theorem T55: Let I be an interval and $f: I \to \mathbb{R}$ a strictly monotone function. Then $f^{-1}: f(I) \to \mathbb{R}$ exists and is continuous.

Theorem T56: Let r be a rational number and define $f: [0, \infty) \to \mathbb{R}$ be given by $f(x) = x^r$. Then f is continuous.

Theorem T57: Let $f, g: D \to \mathbb{R}$ be functions. The following limit laws hold:

1. $\lim\limits_{x \to x_0} f(x) + g(x) = \lim\limits_{x \to x_0} f(x) + \lim\limits_{x \to x_0} g(x)$
2. $\lim\limits_{x \to x_0} f(x) g(x) = \left(\lim\limits_{x \to x_0} f(x)\right)\left(\lim\limits_{x \to x_0} g(x)\right)$
3. $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)}$ provided that $g(x) \neq 0$ in D.

Theorem T58: If $f: D \to \mathbb{R}, g: U \to \mathbb{R}, x_0 \in D, y_0 := \lim_{x \to x_0} f(x)$ is a limit point of U, and $f(D - \{x_0\}) \subseteq U - \{y_0\}$, then:

$$\lim_{x \to x_0} (g \circ f)(x) = \lim_{x \to y_0} g(x)$$

Theorem T59: Let $f, g: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ both be differentiable. Then $f + g$ and fg are also differentiable, as well as $\frac{f}{g}$ is differentiable with the restricted domain $\{x \in D | g(x) \neq 0\}$.

Theorem T60: Let $f: D \to \mathbb{R}$ where $D \subseteq \mathbb{R}$ be differentiable. Then f is continuous on D.

Theorem T61: Let P be a partition of $[a, b]$ and P_2 be a refinement of P. Then $L(f, P) \leq L(f, P_2)$ and $U(f, P_2) \leq U(f, P)$

Theorem T62: Let P_1 and P_2 be partitions of $[a, b]$. Then $L(f, P_1) \leq U(f, P_2)$

Theorem T63: Let $a < b$ and $f: [a, b] \to \mathbb{R}$ be a function. Then $\underline{\int_a^b f} \leq \overline{\int_a^b f}$.

Theorem T64: Let $n \in \mathbb{N}$. The function $f: \mathbb{R} \to \mathbb{R}$ given by $f(x) = x^n$ is differentiable and:

$$f'(x) = nx^{n-1}$$

Theorem T65: Let I be a neighborhood of x_0 and assume the function $f: I \to \mathbb{R}$ is differentiable at x_0. Then f is continuous at x_0.

Theorem T66: Let I be a neighborhood of x_0 and assume the functions $f, g: I \to \mathbb{R}$ are both differentiable at x_0. The following functions are differentiable, satisfy these equations and are typically known as the sum, product, and quotient rules.
(Must have $g'(x) \neq 0$ on the domain in the last one)

$$(f + g)' = f' + g'$$
$$(fg)' = f'g + fg'$$
$$\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$$

Theorem T67: Let $n \in \mathbb{Z}$. The function $f: (0, \infty) \to \mathbb{R}$ given by $f(x) = x^n$ is differentiable and:

$$f'(x) = nx^{n-1}$$

Theorem T68 (Chain Rule): Let I be a neighborhood of x_0. Assume that $f: I \to \mathbb{R}$ is differentiable at x_0 and $g: J \to \mathbb{R}$ is differentiable at $f(x_0)$. Here $J \supseteq f(I)$. Then the function $g \circ f: I \to \mathbb{R}$ is differentiable at x_0 and:

$$(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$$

Lemma L69: Let I be a neighborhood of x_0 and assume $f: I \to \mathbb{R}$ is differentiable at x_0. If x_0 is a maximizer or a minimizer, then $f'(x_0) = 0$.

Theorem T70 (Rolle's Theorem): Let $f: [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). If $f(a) = f(b)$, then there is a point $x_0 \in (a, b)$ such that $f'(x_0) = 0$

Theorem T71 (Mean Value Theorem): Let $f: [a, b] \to \mathbb{R}$ be continuous and differentiable on (a, b). Then there is a point $x_0 \in (a, b)$ such that $f'(x_0) = \frac{f(b)-f(a)}{b-a}$.

Lemma L72: Let $f: I \to \mathbb{R}$ be a differentiable function. Then f is a constant function if and only if $f'(x) = 0$ on I.

Corollary C73: Let I be an open interval and $f, g: I \to \mathbb{R}$ be differentiable functions. f and g differ by a constant if and only if $g'(x) = h'(x)$.

Corollary C74: Let I be an open interval and $f: I \to \mathbb{R}$ be a differentiable function. If $f'(x) > 0$ on I, then f is strictly increasing.

Theorem T75 (2nd Derivative Test): Let I be an open interval an d$f: I \to \mathbb{R}$ be a twice-differentiable function.
If $f'(x_0) = 0$ and $f''(x_0) > 0$, then x_0 is a local minimizer.
If $f'(x_0) = 0$ and $f''(x_0) < 0$, then x_0 is a local maximizer.

Theorem T76: Let $f: [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \dots, x_n\}$ a partition of $[a, b]$. Let m be a lower bound for f and M an upper bound. Then:

$$m(b-a) \leq L(f, P) \leq U(f, P) = M(b-a)$$

Theorem T77: Let $f: [a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, \dots, x_n\}$ a partition of $[a, b]$. Let P^* be a refinement of P. Then:

$$L(f, P) \leq L(f, P^*) \leq U(f, P^*) \leq U(f, P)$$

Theorem T78: Let $f: [a, b] \to \mathbb{R}$ be a bounded function and P_1 and P_2 different partitions of $[a, b]$. Then:

$$L(f, P_1) \leq U(f, P_2)$$

Theorem T79: Let $f: [a, b] \to \mathbb{R}$. The <u>lower integral</u> of f on $[a, b]$ is:

$$\int_{\underline{a}}^{b} f \coloneqq \sup_P \big(L(f, P)\big)$$

Similarly, the <u>upper integral</u> of f is:

$$\int_{a}^{\overline{b}} f \coloneqq \inf_P \big(U(f, P)\big)$$

Theorem T80: Let $f: [a, b] \to \mathbb{R}$.

$$\int_{\underline{a}}^{b} f \leq \int_{a}^{\overline{b}} f$$

Lemma L81: Let $f: [a, b] \to \mathbb{R}$ be bounded and P a partition of $[a, b]$. Then:

$$L(f, P) \leq \underline{\int_a^b} f \leq \overline{\int_a^b} f \leq U(f, P)$$

Theorem T82: Let $f: [a, b] \to \mathbb{R}$ be bounded. Then f is integrable if and only if there is a sequence of partitions $\{P_n\}$ such that:

$$\lim_{n \to \infty} U(f, P_n) - L(f, P_n) = 0$$

Theorem T83: Let $f: [a, b] \to \mathbb{R}$ be a bounded and monotone function. Then f is integrable.

Theorem T84: Let $f: [a, b] \to \mathbb{R}$ be a piecewise constant function (step function). Then f is integrable.

Theorem T85: Let $f: [a, b] \to \mathbb{R}$ be integrable. Then:

$$\int_a^b f = \int_a^c f + \int_c^b f$$

Theorem T86: Let $f, g: [a, b] \to \mathbb{R}$ be integrable and assume that $f(x) \leq g(x)$ for all x. Then:

$$\int_a^b f = \int_a^b g$$

Theorem T87: Let $f, g: [a, b] \to \mathbb{R}$ be integrable. Then:

$$\int_a^b \alpha f + \beta g = \alpha \int_a^b f + \beta \int_a^b g$$

Theorem T88: In addition to the conditions above, let P be a partition of $[a, b]$. Then:

$$L(f, P) + L(g, P) \leq L(f + g, P)$$
$$U(f + g, P) \leq U(f, P) + U(g, P)$$

Theorem T89: Let $f: [a, b] \to \mathbb{R}$ be integrable and assume $|f|$ is too. Then:

$$\left| \int_a^b f \right| \leq \int_a^b |f|$$

Lemma L90: Let $f: [a, b] \to \mathbb{R}$ be continuous and P be a partition of $[a, b]$. Then there are points u, v in a single part of the partition such that:

$$U(f, P) - L(f, P) \leq \big(f(v) - f(u)\big)(b - a)$$

Theorem T91: Let $f: [a, b] \to \mathbb{R}$ be continuous. Then f is integrable.

Theorem T92: Let $f: [a, b] \to \mathbb{R}$ be a function that is continuous on (a, b). Then f is integrable, and $\int_a^b f$ does not depend on $f(a)$ nor $f(b)$.